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What Does Pasting Manage in OMPs?

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A generalized pasting method for orthomodular structures by Navara and Rogalewicz (1991) is sketched. Many applications are presented. It is demonstrated that the method enables to prove valuable nontrivial results.

1. INTRODUCTION

Pasting of Boolean algebras according to Greechie (1971) appeared to be a powerful tool in generating examples of orthomodular lattices (OMLs) and orthomodular posets (OMPs). However, this method has very strict limitations: the Boolean algebras (blocks to be) have to be atomistic, and their intersections can contain, next to minimal and maximal elements, only one atom and its coatom. Other "pasting" methods were introduced by Krausser (1982) and Dichtl (1984). Kalmbach (1984) summarized and slightly generalized all these methods in her monograph.

Later we proved that any OMP can be considered as pasting of its blocks in Dichtl's sense (Rogalewicz, 1988). Nonetheless, Dichtl's method has never been really utilized, probably because of its difficult and nontransparent way of verification of the assumptions. In order to overcome some of these difficulties, we developed new pasting rules and completed them with a new method of atom substitutions (Navara and Rogalewicz, 1991). The new features of our method are:

- Not only blocks, but prefabricated simpler OMPs and/or OMLs can be pasted together; thus, many conditions are trivially satisfied because of their validity in the original OMP.

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- "Continuous" pasting is possible, i.e., the original blocks or posets need not be atomistic, and the intersection can consist of infinitely (even uncountable) many elements.
- The conditions are given also for σ-orthocomplete OMPs, so that the result of pasting can be σ-orthocomplete.
- Projective limits are presented as a type of pasting.
- Any atom can be substituted by another OMP.

In this note, I would like to point out several successful applications of the results of Navara and Rogalewicz (1991).

2. BASIC NOTIONS AND RESULTS

Let L be an OMP. For a detailed introduction, see, e.g., Kalmbach (1984) or Pták and Pulmannová (1991). If $a, b \in L$, $a \leq b$, we put $[a, b]_L = \{c \in L | a \leq c \leq b\}$, and call $[a, b]_L$ an interval in L. The following definition plays the central role in pasting:

Definition. Let $\mathscr{L}$ be a family of OMPs such that all $P, Q \in \mathscr{L}$, $P \neq Q$, satisfy the following conditions:

(Q1) $P \not\subset Q$.
(Q2) $P \cap Q$ is a sub-OMP of both P and Q, and the partial orderings and the orthocomplementations of P and Q coincide on $P \cap Q$.

Let us endow the set $L = \bigcup_{M \in \mathscr{L}} M$ with the relation $\leq_L$ and the unary operation ${}'^L$ defined as follows: $a \leq_L b$ ($a = b'^L$, resp.) if and only if $a \leq_M b$ ($a = b'^M$, resp.) for some $M \in \mathscr{L}$. The set L with $\leq_L$, ${}'^L$ is called the pasting of the family $\mathscr{L}$.

Now we can introduce our main results. The first theorem presents the necessary and sufficient conditions for a pasting of a family of OMPs so that it would be an OMP. These conditions are modified in the next theorem to a form suitable for applications. Then we present a supplementary condition guaranteeing that the pasting results in an OML. Finally, the substitution of an atom is formulated explicitly. All proofs are to be found in Navara and Rogalewicz (1991).

Theorem. The pasting L of a family $\mathscr{L}$ of OMPs is an OMP if and only if the following two conditions hold:

(R1) The relation $\leq_L$ is transitive.
(R2) If $a \perp_M b$ for $M \in L$, then $a \vee_M b = a \vee_L b$.

Theorem. Let *L be the pasting of a family* $\mathscr{L}$ of OMPs. Let $\mathscr{L}$ satisfy the following conditions:

(S1) For each $P, Q \in \mathscr{L}$ and for each $a \in P \cap Q$ there is a $T \in \mathscr{L}$ such that $[0, a]_P \cup [0, a']_Q \subset T$.
(S2) If $a, b, c \in L$ are pairwise orthogonal (in L), then there is an $M \in \mathscr{L}$ such that $\{a, b, c\} \subset M$.

Then L is an OMP.

Theorem. Let the pasting L of a family $\mathscr{L}$ of OMLs be an orthocomplete OMP. Then the following condition is necessary and sufficient for L to be an OML:

(L2) If $c, d \in L$ are any upper bounds of $a, b \in L$, then there is an upper bound e of both a and b such that $e \leq c$, $e \leq d$.

Theorem. Let K, L be OMPs and let a be an atom in K. We denote by b the coatom a'^K and by M the product $[0, b]_K \times L$. For all $c \in [0, b]_K$ we unify c ($\in K$) with $(c, 0_L) \in M$ and $c \vee_K a \in K$ with $(c, 1_L) \in M$. We endow the union $P = K \cup M$ with the partial ordering equal to the union of the ordering of K and M. The orthocomplementation on P is defined as follows:

(i) If $c \in K$, then $c'^P = c'^K$.
(ii) If $c \in M - K$, then c is of the form $(d, e) \in M$ and $c'^P = c'^M = (d'^K \wedge_K e'^L) \in M$.

Then P is an OMP. We say that P originated in the substitution of the atom a in K with the OMP L.

3. APPLICATIONS

3.1. Examples of OMPs with Particular Properties

This is just a selection of such examples:

(a) A finite OML in which every block has the same cardinality k is called λ-regular if each atom is a member of just λ blocks. Being given $\lambda \in \mathrm{N}$ and $k \geq 4$, we construct a λ-regular OML with λ^2 blocks (Rogalewicz, 1989). This improves the estimation of cardinality of such OMLs (Köhler, 1982) qualitatively.

(b) There exists an infinite OMP with a finite set of generators, the sublogics of which are only concrete (=set representable) logics (Rogalewicz, 1991). (On the other hand, finitely generated concrete logics are finite.)

(c) Orthosymmetric OMLs were introduced by Mayet (1991). Modular lattices need not admit an orthosymmetric structure, and orthosymmetric OMLs need not have a strong set of states (Hamhalter and Navara, 1991).

(d) There is an atomistic OML L such that each atom is noncompatible only with a finite number of atoms, but L cannot be expressed as a product of simpler OMLs (Pulmannová and Rogalewicz, 1991).

(e) There exist OMLs admitting no nontrivial group-valued measure (Navara, 1992*c*).

3.2. Logics with Predetermined Centers, State Spaces, and Automorphism Groups

Let K be an OMP whose state space is nonvoid. Let B be a Boolean algebra, C a compact convex subset of a locally convex topological linear space, and G a group. Then K can be enlarged to an OMP L such that the center of L equals B, the state space of L equals C, and the group of automorphisms equals G (Navara *et al.*, 1988; Navara, 1992*b*). The result remains valid if we substitute the abbreviation "OML" for "OMP." Generalized pasting is the main tool in the construction of L; moreover, these constructions are impossible in the Greechie's or Dichtl's approaches.

Every concrete OMP can be enlarged to a concrete logic with a given automorphism group and a given center (Navara and Tkadlec, 1991). Since not every state space of a (general) OMP is affinely homeomorphic to the state space of a concrete logic, the assumption concerning the state space is not possible in concrete OMPs. In σ-orthocomplemented OMPs or OMLs, every OMP (with a nonvoid state space) can be enlarged to an OMP with an arbitrary center and an arbitrary state space under the condition that the center admits at least one two-valued state (Navara and Pták, 1982).

These results show that the pasting techniques can be used not only to generate a series of examples and counterexamples, but also to help in proving deep positive results.

3.3. Description of the State Space of an OML

One of the first utilizations of the pastings due to Greechie (1971) was the result by Shultz (1974) characterizing the state space of a (finitely additive) OML: Each compact convex set is affinely homeomorphic to the state space of an OML. Using our generalized pasting technique, Navara (1992*a*) gave a much simpler proof of this theorem.

Moreover, Navara and Rüttimann (1991) found a similar theorem for the σ-additive case. They introduced the notion of s-semiexposed faces of a compact convex subset of a locally convex Hausdorff topological vector space. They showed that every s-semiexposed face of any compact convex set is affinely homeomorphic to the σ-state space of an OMP. Generalized pasting plays an important role in the proof.

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